

# Intermittency transition to generalized synchronization in coupled time-delay systems

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We report the nature of the transition to generalized synchronization (GS) in a system of two coupled scalar piecewise linear time-delay systems using the auxiliary system approach. We demonstrate that the transition to GS occurs via an on-off intermittency route and that it also exhibits characteristically distinct behaviors for different coupling configurations. In particular, the intermittency transition occurs in a rather broad range of coupling strength for the error feedback coupling configuration and in a narrow range of coupling strength for the direct feedback coupling configuration. It is also shown that the intermittent dynamics displays periodic bursts of periods equal to the delay time of the response system in the former case, while they occur in random time intervals of finite duration in the latter case. The robustness of these transitions with system parameters and delay times has also been studied for both linear and nonlinear coupling configurations. The results are corroborated analytically by suitable stability conditions for asymptotically stable synchronized states and numerically by the probability of synchronization and by the transition of sub-Lyapunov exponents of the coupled time-delay systems. We have also indicated the reason behind these distinct transitions by referring to the unstable periodic orbit theory of intermittency synchronization in low-dimensional systems.

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## I. INTRODUCTION

Synchronization of interacting chaotic oscillators is one of the most interesting nonlinear phenomenon and is an inherent part of many natural systems (cf. [1,2]). The concept of synchronization is receiving a central importance in recent research in nonlinear dynamics due to its potential applications in diverse areas of science and technology. Since the identification of chaotic synchronization [3–5], several works have appeared in identifying and demonstrating basic kinds of synchronization both theoretically and experimentally [1–5]. There are also attempts to find a unifying framework for defining the overall class of chaotic synchronizations [6–8].

One of the interesting synchronization behaviors of unidirectionally coupled chaotic systems is generalized synchronization (GS), which was conceptually introduced in Ref. [9]. Generalized synchronization is observed in coupled non-identical systems, where there exists some functional relation between the drive  $X(t)$  and the response  $Y(t)$  systems—that is,  $Y(t)=F(X(t))$ . With GS, all the response systems coupled to the drive lose their intrinsic chaoticity (sensitivity to initial conditions) under the same driving and follow the same trajectory. Hence the presence of GS can be detected using the so-called auxiliary system approach [10], where an additional system (auxiliary system) identical to the response system is coupled to the drive in similar fashion. The auxiliary system approach is particularly appealing since it can be implemented directly in an experiment and, in addition, this method allows one to utilize analytical approaches for studying GS. However, one has to be aware that if there are multiple basins of attraction for the coupled drive-response system, then the auxiliary system approach can fail.

Generalized synchronization has been well studied and understood in systems with few degrees of freedom and for discrete maps [9–15]. The concept of GS has also been extended to spatially extended chaotic systems such as coupled Ginzburg-Landau equations [16]. Recently, the terminology intermittent generalized synchronization (IGS) [17] was introduced in diffusively coupled Rössler systems, in analogy with intermittent lag synchronization (ILS) [18,19] and intermittent phase synchronization (IPS) [20–22], and also experimentally in coupled Chua’s circuits. Very recently, it has been shown [23] that the transition to intermittent chaotic synchronization (in the case of complete synchronization) is characteristically distinct for geometrically different chaotic attractors. In particular, it has been shown that for phase-coherent chaotic attractors (Rössler attractor) the transition occurs immediately as soon as the coupling strength is increased from zero, and for nonphase-coherent attractors (Lorenz attractor), the transition occurs slowly as the coupling is increased from zero.

Time-delay systems form an important class of dynamical systems and recently they are receiving central importance in investigating the phenomenon of chaotic synchronizations in view of their infinite-dimensional nature and feasibility of experimental realization [24–27]. While the concept of GS has been well established in low-dimensional systems, it has not yet been studied in detail in coupled time-delay systems and only very few recent studies have dealt with GS in time-delay systems [24,25]. In particular, the mechanism of onset of GS in coupled time-delay systems and its characteristic properties have not yet been clearly understood and require urgent attention.

In this paper, we investigate the characteristic properties of the nature of the onset of GS from an asynchronous state in unidirectionally coupled piecewise linear time-delay systems exhibiting highly nonphase-coherent hyperchaotic attractors [26]. We find that the onset of GS is preceded by an on-off intermittency mechanism from the desynchronized state. We have also identified that the intermittency transition

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to GS exhibits characteristically distinct behaviors for different coupling schemes. In particular, the intermittency transition occurs in a broad range of coupling strength for error feedback coupling configurations and in a narrow range of coupling strength for direct feedback coupling configurations, beyond a certain threshold value of coupling strength. In addition, the intermittent dynamics is characterized by periodic bursts away from the temporal synchronized state with period equal to the delay time of the response system in the case of a broad range intermittency transition whereas it is characterized by random time intervals in the case of a narrow range intermittency transition. We have also confirmed these dynamical behaviors in both linear and nonlinear coupling configurations. We have analyzed these transitions analytically using a Krasvoskii-Lyapunov functional approach and numerically by the probability of synchronization and by the sub-Lyapunov exponents. We have also addressed the reason behind these transitions using periodic orbit theory. The robustness of these transitions with the system parameters, in both the linear and nonlinear, error feedback and direct feedback coupling configurations, is also studied.

The plan of the paper is as follows. In Sec. II, we will point out the existence of a broad range intermittency transition to GS for linear error feedback coupling proportional to  $x_1(t) - x_2(t)$  while in Sec. III the existence of a narrow range intermittency transition is shown for linear direct feedback coupling of the form  $x_1(t)$ , where  $x_1(t)$  and  $x_2(t)$  are the drive and response signals, respectively (see below for details). In Sec. IV we will discuss the existence of a broad range intermittency transition for nonlinear error feedback coupling of the form  $f(x_1(t - \tau_2)) - f(x_2(t - \tau_2))$ , where  $f(x)$  is an odd piecewise linear function. The existence of a narrow range intermittency transition is discussed in Sec. V for nonlinear direct feedback coupling of the form  $f(x_1(t - \tau_2))$ . Finally in Sec. VI, we will summarize our results.

**II. BROAD RANGE (SLOW OR DELAYED) INTERMITTENCY TRANSITION TO GS FOR LINEAR ERROR FEEDBACK COUPLING OF THE FORM  $x_1(t) - x_2(t)$**

We consider the following first-order delay differential equation introduced by Lu and He [28] and discussed in detail by Thangavel *et al.* [29],

$$\dot{x}(t) = -ax(t) + bf(x(t - \tau)), \tag{1}$$

where  $a$  and  $b$  are parameters,  $\tau$  is the time-delay, and  $f$  is an odd piecewise linear function defined as

$$f(x) = \begin{cases} 0, & x \leq -4/3, \\ -1.5x - 2, & -4/3 < x \leq -0.8, \\ x, & -0.8 < x \leq 0.8, \\ -1.5x + 2, & 0.8 < x \leq 4/3, \\ 0, & x > 4/3. \end{cases} \tag{2}$$

Recently, we have reported [30] that the system (1) exhibits hyperchaotic behavior for suitable parametric values. For our present study, we find that for the choice of the parameters

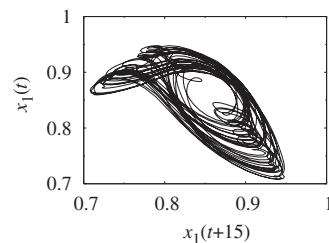


FIG. 1. The hyperchaotic attractor of the system (1) for the parameter values  $a=1.0$ ,  $b=1.2$ , and  $\tau=15.0$ .

$a=1.0$ ,  $b=1.2$ , and  $\tau=15.0$  with the initial condition  $x(t) = 0.9$ ,  $t \in (-5, 0)$ , Eq. (1) exhibits hyperchaos. The corresponding pseudoattractor is shown in Fig. 1. The hyperchaotic nature of Eq. (1) is confirmed by the existence of multiple positive Lyapunov exponents. The first ten maximal Lyapunov exponents for the above choice of parameters as a function of delay time  $\tau$  are shown in Fig. 2 (the spectrum of Lyapunov exponents in this paper are calculated using the procedure suggested by Farmer [31]).

To be specific, we first consider the following unidirectional, linearly coupled systems with drive  $x_1(t)$ , response  $x_2(t)$ , and an auxiliary  $x_3(t)$ :

$$\dot{x}_1(t) = -ax_1(t) + b_1f(x_1(t - \tau_1)), \tag{3a}$$

$$\dot{x}_2(t) = -ax_2(t) + b_2f(x_2(t - \tau_2)) + b_3[x_1(t) - x_2(t)], \tag{3b}$$

$$\dot{x}_3(t) = -ax_3(t) + b_2f(x_3(t - \tau_2)) + b_3[x_1(t) - x_3(t)], \tag{3c}$$

where  $b_1$ ,  $b_2$ , and  $b_3$  are constant parameters and  $\tau_1$  and  $\tau_2$  are constant delay parameters. Note that when  $b_1 \neq b_2$  or  $\tau_1 \neq \tau_2$  or both, corresponding to parameter mismatches, we have unidirectionally coupled nonidentical systems [Eqs. (3a) and (3b)], while the auxiliary system is given by (3c) and  $f(x)$  is the odd piecewise linear function (2). The cou-

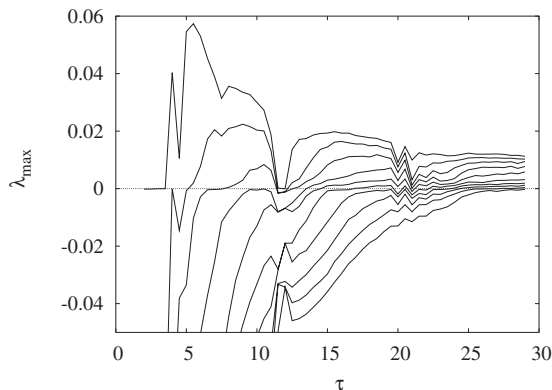


FIG. 2. The first ten maximal Lyapunov exponents  $\lambda_{\max}$  of the scalar time-delay system for the parameter values  $a=1.0$ ,  $b_1=1.2$ , and  $\tau \in (2, 29)$  [which is the same as Eq. (1) with  $b_1$  replaced by  $b$ ], Eq. (3a).

pling in (3b) may be also called a linear error feedback coupling.

For simplicity, we have chosen  $b_1=b_2$  so that the time delays  $\tau_1$  and  $\tau_2$  alone introduce a simple form of parameter mismatch between the drive  $x_1(t)$  and the response  $x_2(t)$ . We have chosen the values of the parameters as  $a=1.0$ ,  $b_1=b_2=1.2$ ,  $\tau_1=20$ , and  $\tau_2=25$ . For this parametric choice, in the absence of coupling, all three systems (3a)–(3c) evolve independently and exhibit hyperchaotic attractors, which is confirmed by the existence of multiple positive Lyapunov exponents (Fig. 2). The actual value of the positive Lyapunov exponents for  $\tau=20$  are 0.009 16, 0.007 59, 0.005 65, 0.002 83, and 0.000 73 and for  $\tau=25$  they are 0.012 34, 0.010 67, 0.008 86, 0.006 58, 0.003 86, 0.002 29, 0.001 23, and 0.000 33.

### A. Stability condition

With GS, as all response systems under the same driving follow the same trajectory, it is sufficient to identify the existence condition for establishment of complete synchronization (CS) between the response  $x_2(t)$  and the auxiliary  $x_3(t)$  systems in order to achieve GS between the drive  $x_1(t)$  and the response  $x_2(t)$  systems.

Now, for CS to occur between the response  $x_2(t)$  and auxiliary  $x_3(t)$  variables, we consider the time evolution of the difference system with the state variable  $\Delta=x_3(t)-x_2(t)$ . It can be written for small values of  $\Delta$  as

$$\dot{\Delta} = -(a + b_3)\Delta + b_2 f'(x_2(t - \tau_2))\Delta_{\tau_2}, \quad (4)$$

where

$$f'(x) = \begin{cases} -1.5, & -4/3 < x \leq -0.8, \\ 1, & -0.8 < x \leq 0.8, \\ -1.5, & 0.8 < x \leq 4/3. \end{cases} \quad (5)$$

The synchronization manifold  $x_2(t)=x_3(t)$  is locally attracting if the origin of this equation is stable. Following a Krasovskii-Lyapunov functional approach, we define a positive definite Lyapunov functional of the form [27,32,33] (details of the stability analysis are given in the Appendix)

$$V(t) = \frac{1}{2}\Delta^2 + \mu \int_{\tau_2}^0 \Delta^2(t + \theta) d\theta, \quad (6)$$

where  $\mu$  is an arbitrary positive parameter,  $\mu > 0$ . The solution of Eq. (4)—namely,  $\Delta=0$ —is stable if the derivative of the functional along the trajectory of Eq. (4) is negative. This negativity condition is satisfied if  $b_3 + a > \frac{b_2^2 f'^2(x_2(t - \tau_2))}{4\mu} + \mu$ , from which it turns out that the sufficient condition for asymptotic stability is

$$a + b_3 > |b_2 f'(x_2(t - \tau_2))|. \quad (7)$$

Now from the form of the piecewise linear function  $f(x)$  given by Eq. (2), we have

$$|f'(x_2(t - \tau_2))| = \begin{cases} 1.5, & 0.8 \leq |x_2| \leq 4/3, \\ 1.0, & |x_2| < 0.8. \end{cases} \quad (8)$$

Consequently the stability condition (7) becomes  $a + b_3 > |1.5b_2| > |b_2|$ . Thus one can take

$$a + b_3 > |b_2| \quad (9)$$

as the less stringent or approximate stability condition [as the synchronization dynamics of the coupled systems (3a)–(3c) can occur even beyond the inner region,  $|x_2| < 0.8$ ] for (7) to be valid, while

$$a + b_3 > |1.5b_2| \quad (10)$$

can be considered as the most general or stringent or exact stability condition [as the full synchronization dynamics of the coupled systems (3a)–(3c) lies within the outer region  $0.8 < |x_2| \leq \frac{4}{3}$ ] specified by (7) for asymptotic stability of the synchronized state  $\Delta=0$ .

### B. Approximate (intermittent) generalized synchronization

In order to understand the mechanism of the transition to a synchronized state, it will be important to follow the dynamics from the parameter values at which the less stringent condition is satisfied. Figure 3(a) shows the approximate GS (which may also be termed as IGS in analogy with ILS) between the drive  $x_1(t)$  and the response  $x_2(t)$  systems, whereas Fig. 3(b) shows the approximate CS between the response  $x_2(t)$  and the auxiliary  $x_3(t)$  systems for the values of the parameters  $a=1.0$ ,  $b_1=b_2=1.2$ ,  $\tau_1=20$ ,  $\tau_2=25$ , and  $b_3=0.4$  satisfying the less stringent condition (9). Perfect GS and perfect CS are shown in Figs. 3(c) and 3(d), respectively, for  $b_3=0.9$  satisfying the general stability condition (10). Time traces of the difference  $x_2(t)-x_3(t)$  corresponding to approximate CS [Fig. 3(b)] are shown in Fig. 4, which show periodic bursts with a period between two consecutive bursts approximately equal to the time delay of the response system,  $t \approx 25$ , when “on” states of amplitude greater than  $|0.01|$  are considered. Figure 4(b) shows an enlarged (in  $x$  scale) part of Fig. 4(a) to view the bursts at periodic intervals when bursts of larger amplitudes ( $\Delta > |0.01|$ ) are considered, while Fig. 4(c) is an enlarged (in  $y$  scale) version of Fig. 4(b) to show random bursts when bursts of smaller amplitude  $\Delta < |0.01|$  are considered.

Usually the intermittent dynamics is characterized by the entrainment of the dynamical variables in random time intervals of finite duration [34,35]. But from Fig. 4(b), it is evident that the intermittent dynamics displays periodic bursts from the synchronous state with period approximately equal to the delay time of the response system, when amplitudes of the state variable  $|\Delta|=|x_3(t)-x_2(t)| > 0.01$  are considered, for the values of the coupling strength at which the less stringent stability condition (9) is satisfied. The statistical features associated with the intermittent dynamics are analyzed by calculating the distribution of laminar phases  $\Lambda(t)$  with amplitude less than a threshold value of  $\Delta$ . A universal asymptotic

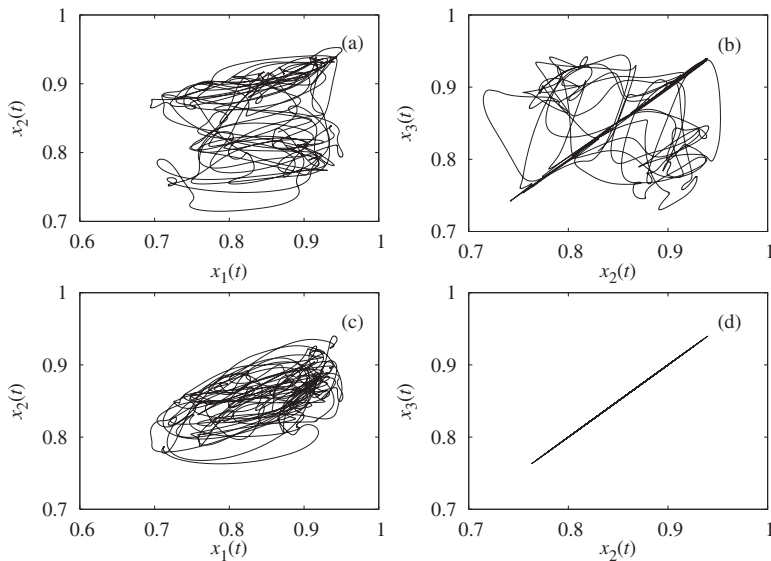


FIG. 3. Dynamics in the phase space of the systems (3a)–(3c). (a) and (b) Approximate GS and CS, respectively, for the value of the coupling strength  $b_3=0.4$ . (c) and (d) Perfect GS and CS, respectively, for the value of the coupling strength  $b_3=0.9$ .

power law distribution  $\Lambda(t) \propto t^{-\alpha}$  is observed for the threshold value  $\Delta=|0.0001|$  with the value of the exponent  $\alpha=-1.5$  as shown in Fig. 5(b), which is quite typical for on-off intermittency. On the other hand, the distribution of laminar phases  $\Lambda(t)$  for the value of the threshold value

of  $|\Delta=0.1|$  shows a periodic structures [Fig. 5(a)], where the peaks occur approximately at  $t=nT$ ,  $n=1, 2, \dots$ , where  $T \approx \tau_2$  is the period of the lowest periodic orbit of the uncoupled system (3b). Note that a  $-3/2$  power law is observed for the intermittent dynamics shown in Fig. 4 for laminar phases  $\Lambda(t)$  with amplitude less than  $\Delta=|0.0001|$  (as an illustrative example), which is also evident from Fig. 4(c), while periodic bursts are observed for “on” state of amplitude greater than  $|0.01|$ . It is to be noted that such periodic bursts of period approximately equal to the time delay of the response system for a larger threshold value of  $\Delta$  along with the on-off intermittency behavior have also been observed by Zhan *et al.* [24] in unidirectionally coupled Mackey-Glass systems, where the authors discussed the relation between two modes of synchronization—namely, CS and GS.

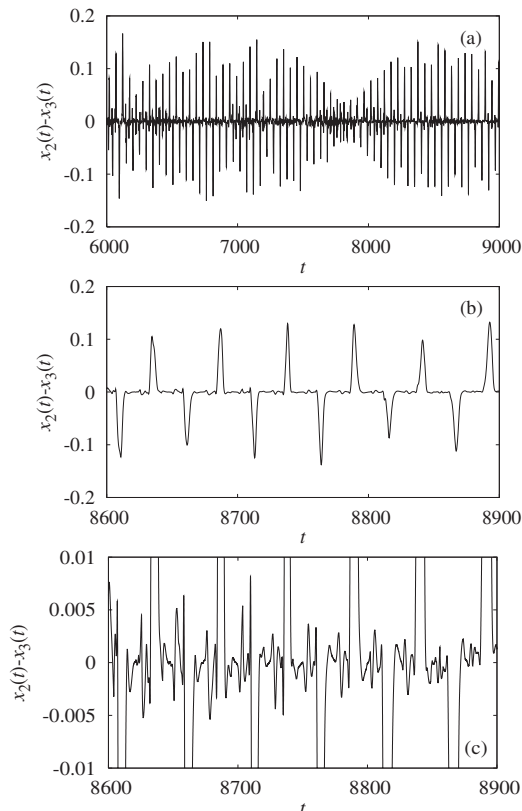


FIG. 4. The intermittent dynamics of the response  $x_2(t)$  and auxiliary  $x_3(t)$  systems for the value of the coupling strength  $b_3=0.4$ . (a) Time traces of the difference  $x_2(t)-x_3(t)$  corresponding to Fig. 3(b). (b) Enlarged in  $x$  scale to show bursts at periodic intervals when bursts of larger amplitudes  $\Delta > |0.01|$  are considered. (c) Enlarged in  $y$  scale to show random bursts when bursts of smaller amplitudes  $\Delta < |0.01|$  are considered.

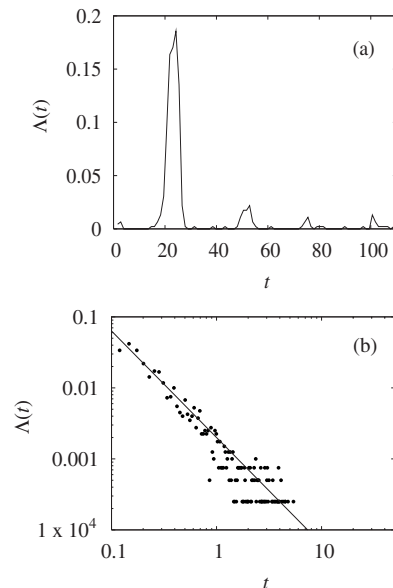


FIG. 5. The statistical distribution of laminar phases corresponding to Fig. 4. (a) For threshold value  $\Delta=|0.1|$  and (b) for the threshold value of  $\Delta=|0.0001|$ .

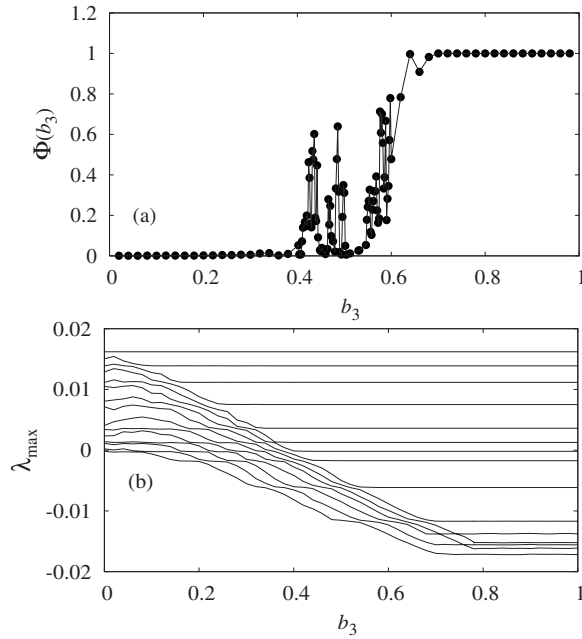


FIG. 6. (a) The probability of synchronization,  $\Phi(b_3)$ , between the response  $x_2(t)$  and auxiliary  $x_3(t)$  systems. (b) Largest Lyapunov exponents of the coupled drive  $x_1(t)$  and response  $x_2(t)$  systems (3a) and (3b).

### C. Characterization of IGS

Now we characterize the intermittency transition to GS by using (i) the notion of the probability of synchronization,  $\Phi(b_3)$ , as a function of the coupling strength  $b_3$  [23], which can be defined as the fraction of time during which  $|x_2(t) - x_3(t)| < \varepsilon$  occurs, where  $\varepsilon$  is a small but arbitrary threshold, and (ii) from the changes in the sign of sub-Lyapunov exponents (which are nothing but the Lyapunov exponents of the subsystem) in the spectrum of Lyapunov exponents of the coupled time-delay systems. Figure 6(a) shows the probability of synchronization,  $\Phi(b_3)$ , as a function of the coupling strength  $b_3$  calculated from the variables of the response  $x_2(t)$  and the auxiliary  $x_3(t)$  systems for CS between them. For the range of  $b_3 \in (0, 0.39)$ , there is the absence of any entrainment between the systems resulting in asynchronous behavior and the probability of synchronization,  $\Phi(b_3)$ , is practically zero in this region. However, starting from the value of  $b_3 = 0.39$  and above, there appear oscillations in the value of the probability of synchronization  $\Phi(b_3)$  between zero and some finite values less than unity, exhibiting an intermittency transition to GS in the range of  $b_3 \in (0.4, 0.62)$  for which the less stringent stability condition (9) is satisfied. Beyond  $b_3 = 0.62$ ,  $\Phi(b_3)$  attains unit value, indicating perfect GS. Note that the above intermittency transition occurs in a rather wide range of the coupling strength (this can also be termed as a slow or delayed intermittency transition in analogy with the terminology used in [23]), which has also been confirmed from the transition of successive largest sub-Lyapunov exponents in the corresponding range of coupling strength.

The spectra of the first 15 largest Lyapunov exponents  $\lambda_{\max}$  of the coupled drive  $x_1(t)$  and response  $x_2(t)$  systems are shown in Fig. 6(b). From the general stability condition (7),

it is evident that for the chosen value of the parameter  $a = 1.0$ , the less stringent stability condition (9) is satisfied for the values of coupling strength  $b_3 > 0.2$ . Correspondingly, the least positive sub-Lyapunov exponent of the response system (3b) gradually becomes negative from  $b_3 > 0.2$ . Subsequently, the remaining positive sub-Lyapunov exponents gradually become negative and attain saturation in the range of  $b_3 \in (0.2, 0.8)$ . This is in accordance with the fact that the less stringent stability condition is satisfied only in the corresponding range of coupling strength  $b_3$ . This is a strong indication of the broad range intermittency (IGS) transition to GS. For  $b_3 > 0.8$ , the general stability condition (10) is satisfied, where one can observe perfect GS as is evidenced both by the probability of synchronization approaching unit value and by the negative saturation of sub-Lyapunov exponents calculated between the drive and response systems. The inference is that the correlation between the oscillations of the systems eventually becomes stronger with the strength of the coupling, and this is indicated by the successive transition of sub-Lyapunov exponents to negative values.

It is a well-established fact that a chaotic attractor can be considered as a pool of infinitely many unstable periodic orbits (UPOs) of all periods. Synchronization between two coupled systems is said to be asymptotically stable if all the unstable periodic orbits of the response system are stabilized in the transverse direction of the synchronization manifold. Consequently, all the trajectories transverse to the synchronization manifold converge for suitable values of coupling strength and this is reflected in the negative values of the transverse Lyapunov exponents (sub-Lyapunov exponents) upon synchronization [23]. From our results, we find that the sub-Lyapunov exponents gradually become negative in a broad range of coupling strength  $b_3$  after a certain threshold value and this is in accordance with the known results on gradual stabilization of unstable periodic orbits of the response system in the complex synchronization manifold of low-dimensional systems [23]. Unfortunately, methods for locating UPOs and calculating their transverse Lyapunov exponents have not been well established for time-delay systems and hence a quantitative proof of the gradual stabilization of UPOs has not been given here. However, the gradual stabilization of UPOs along with their transverse Lyapunov exponent in the range of the intermittency transition has been reported for the case of coupled Rössler and Lorenz systems in Ref. [23]. It can then be inferred from these studies that the broad range intermittency transition in the case of error feedback coupling configuration is due to the fact that the strength of the coupling  $b_3$  contributes only less significantly to stabilize the UPOs as the error  $x_1(t) - x_2(t)$  gradually becomes smaller from the transition regime after a certain threshold value of the coupling strength.

The robustness of the intermittency transition in a broad range of coupling strength with the system parameter  $b_2 \in (1.1, 1.6)$  and with the coupling delay  $\tau_2 \in (10, 20)$  has also been confirmed. Figure 7(a) shows the three-dimensional plot of the probability of synchronization as a function of the system parameter  $b_2$  and the coupling strength  $b_3$ , while Fig. 7(b) shows the three-dimensional plot of  $\Phi(b_3)$  as a function of the coupling delay  $\tau_2$  and the coupling strength  $b_3$ . The above figures (Fig. 7) clearly

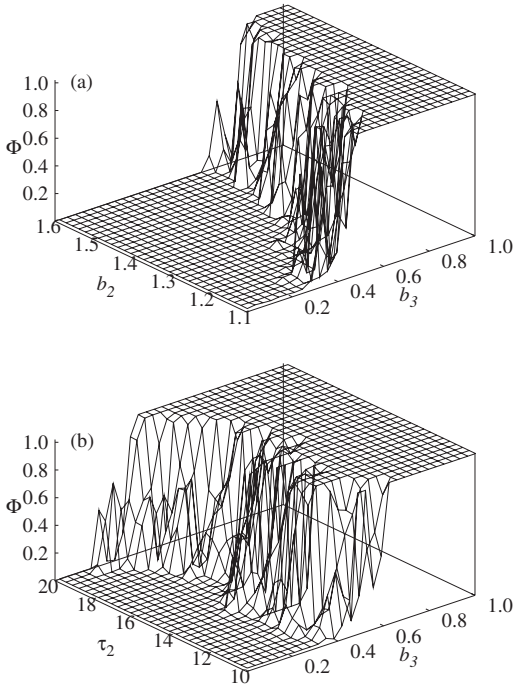


FIG. 7. The probability of synchronization,  $\Phi(b_3)$ , in three-dimensional plots (a) as a function of the system parameter  $b_2$  and the coupling strength  $b_3$  and (b) as a function of the coupling delay  $\tau_2$  and the coupling strength  $b_3$ , exhibiting a broad range intermittency transition to GS for linear error feedback coupling.

reveal the broad range intermittency transition to GS in the case of the linear error feedback coupling scheme.

### III. NARROW RANGE (IMMEDIATE) INTERMITTENCY TRANSITION TO GS FOR LINEAR DIRECT FEEDBACK COUPLING OF THE FORM $x_1(t)$

To illustrate the narrow range intermittency transition to GS, we consider the unidirectional linear direct feedback coupling of the form

$$\dot{x}_1(t) = -ax_1(t) + b_1 f(x_1(t - \tau_1)), \quad (11a)$$

$$\dot{x}_2(t) = -ax_2(t) + b_2 f(x_2(t - \tau_2)) + b_3 x_1(t), \quad (11b)$$

$$\dot{x}_3(t) = -ax_3(t) + b_2 f(x_3(t - \tau_2)) + b_3 x_1(t), \quad (11c)$$

where  $f(x)$  is of the same odd piecewise linear form as in Eq. (2). Assuming the same values of the parameters as before and proceeding in the same way as in the previous case, one can obtain the sufficient condition for asymptotically stable CS between the response  $x_2(t)$ , Eq. (11b), and the auxiliary  $x_3(t)$ , Eq. (11c), systems as

$$a > |b_2 f'(x_2(t - \tau_2))|. \quad (12)$$

It is to be noted that the above stability condition holds good only for the case when coupling is present—that is,  $b_3 \neq 0$ . When there is no coupling ( $b_3 = 0$ ), by definition, there will be a desynchronized chaotic state. As soon as the value of the coupling strength is increased from zero, the stability condition (12) always leads to a synchronized state even for very feeble values of  $b_3$  for parameters satisfying the stability condition, as it is independent of the coupling strength  $b_3$ . As the values of the parameters satisfying the stability condition (12) rapidly leads to an immediate transition to the synchronized state as soon as the coupling is switched on, it is difficult to identify the possible transitions to the synchronized state. In addition, as the stability condition is independent of the coupling strength  $b_3$ , one is not able to explore the dynamical transitions as a function of coupling strength for the parameter values satisfying the stability condition (12). Hence we study the synchronization transition by choosing the parameters violating the stability condition as  $a = 1.0$ ,  $b_1 = 1.2$ ,  $b_2 = 1.1$ , and  $\tau_1 = \tau_2 = 20$  and varying the coupling strength  $b_3$  in order to identify the mechanism of the synchronization transition. Here, in this case  $b_1$  and  $b_2$  alone introduce the parameter mismatch while  $\tau_1 = \tau_2$  (it may be added that the qualitative nature of the dynamical transitions remains the same even when the mismatch is either in time delays alone—that is,  $\tau_1 \neq \tau_2$ ,  $b_1 = b_2$ —or in both the system

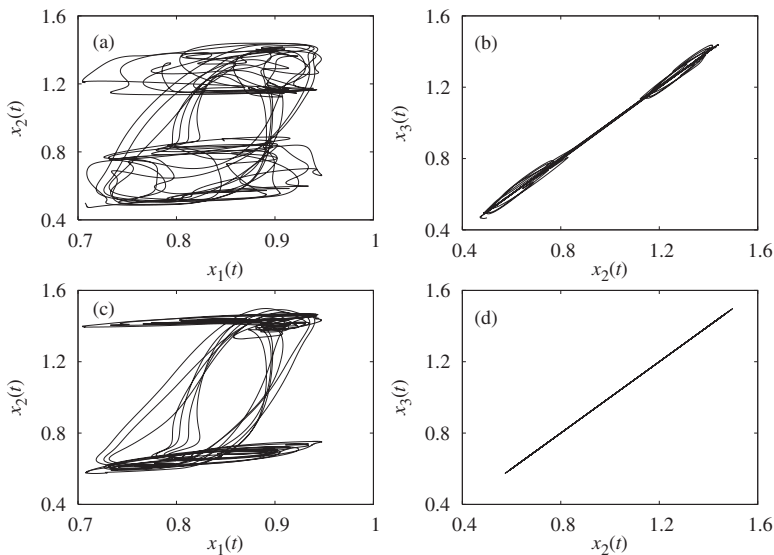


FIG. 8. Dynamics in the phase space of the systems (11a)–(11c): (a) and (b) Approximate GS and CS, respectively, for the value of the coupling strength  $b_3 = 0.64$ . (c) and (d) Perfect GS and CS, respectively, for the value of the coupling strength  $b_3 = 0.8$ .

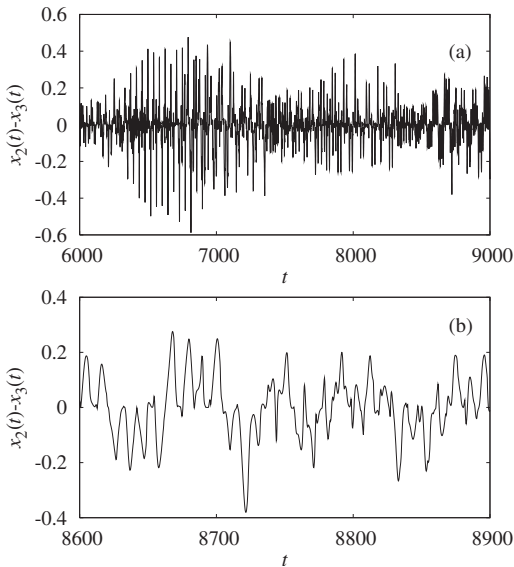


FIG. 9. The intermittent dynamics of the response  $x_2(t)$ , Eq. (11b), and auxiliary  $x_3(t)$ , Eq. (11c), systems for the value of the coupling strength  $b_3=0.64$ . (a) and (b) Time traces of the difference  $x_2(t)-x_3(t)$  corresponding to Fig. 8(b).

parameters and time delays,  $b_1 \neq b_2$ ,  $\tau_1 \neq \tau_2$ , as confirmed below in the three-dimensional plots of Fig. 12.

As  $b_3$  is varied from zero, the transition from the desynchronized state to approximate GS occurs for  $b_3 > 0.6$ . Approximate GS (IGS) between the drive system  $x_1(t)$  specified by Eq. (11a) and the response system  $x_2(t)$  represented by Eq. (11b) is shown in Fig. 8(a) whereas the approximate CS between the response  $x_2(t)$  [Eq. (11b)] and the auxiliary  $x_3(t)$  [Eq. (11c)] systems is shown in Fig. 8(b) for the value of the coupling strength  $b_3=0.64$ . Perfect GS and CS are shown in Figs. 8(c) and 8(d), respectively, for the value of the coupling strength  $b_3=0.8$ . The intermittent dynamics at the transition regime corresponding to the value of the coupling strength  $b_3=0.64$  is shown in Fig. 9, in which Fig. 9(b) shows the enlarged part of Fig. 9(a). It is clear from this figure that the intermittent dynamics displays intermittent bursts at random time intervals. The statistical distribution of the laminar phases again shows a universal asymptotic  $-1.5$  power law behavior for the threshold value  $\Delta=|0.0001|$ , which is typical for on-off intermittency transitions, as shown in Fig. 10.

Now we characterize the intermittency transition to GS in the present case, again by using the notion of the probability

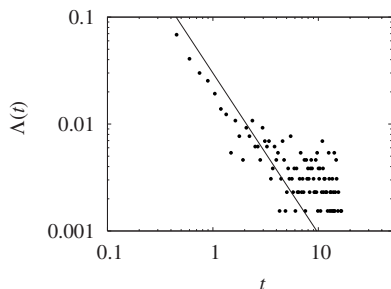


FIG. 10. The statistical distribution of laminar phases for Fig. 9.

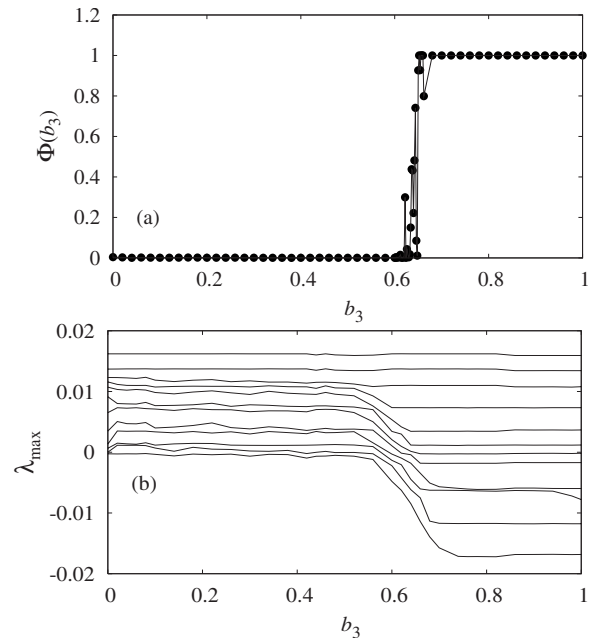


FIG. 11. (a) The probability of synchronization,  $\Phi(b_3)$ , between the response  $x_2(t)$ , Eq. (11b), and the auxiliary  $x_3(t)$ , Eq. (11c), systems. (b) Largest Lyapunov exponents of the coupled drive  $x_1(t)$  and response  $x_2(t)$  systems (11a) and (11b).

of synchronization,  $\Phi(b_3)$ , and from the changes in the sign of sub-Lyapunov exponents of the coupled system. The probability of synchronization is shown in Fig. 11(a) as a function of the coupling strength, again calculated from the response  $x_2(t)$  and the auxiliary  $x_3(t)$  systems, Eqs. (11b) and (11c), respectively, which remains zero in the range of  $b_3 \in (0, 0.60)$  and oscillates between its maximum and minimum values in a narrow range of  $b_3 \in (0.60, 0.68)$ , confirming the existence of approximate CS in the latter range. Above  $b_3=0.68$  the probability of synchronization acquires its maximum value depicting perfect CS between the response  $x_2(t)$  and the auxiliary  $x_3(t)$  systems. Correspondingly there exists perfect GS between the drive  $x_1(t)$  and the response  $x_2(t)$  systems. Figure 11(b) shows the first 12 maximal Lyapunov exponents of the coupled drive  $x_1(t)$  and the response  $x_2(t)$  systems. The least positive sub-Lyapunov exponent of the response system starts to become negative from  $b_3 > 0.60$ . Subsequently, all the other positive sub-Lyapunov exponents become negative and reach saturation in a rather narrow range of  $b_3 \in (0.60, 0.68)$ . Thus the narrow range intermittency (IGS) transition (this can also be termed as the immediate intermittency transition in analogy with the terminology used in [23]) is confirmed from both the probability of synchronization, calculated from the response and auxiliary systems, and negative saturation of sub-Lyapunov exponents, calculated from the drive and response systems.

As discussed in the previous section, the narrow range intermittency transition is in accordance with the stabilization of all unstable periodic orbits of the response system in a narrow range as a function of the coupling strength  $b_3$  and this is reflected in the immediate transition of all the sub-Lyapunov exponents [Fig. 11(b)] to negative values. It is also to be noted that the narrow range intermittency transition in

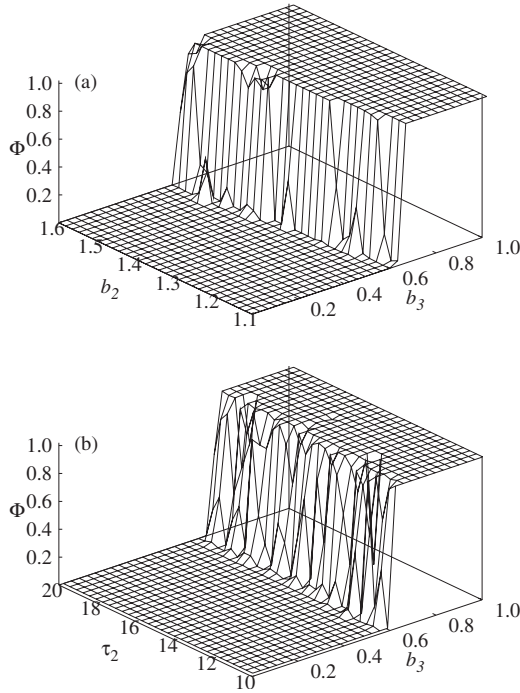


FIG. 12. The probability of synchronization,  $\Phi(b_3)$ , in three-dimensional plots (a) as a function of the system parameter  $b_2$  and the coupling strength  $b_3$  and (b) as a function of the coupling delay  $\tau_2$  and the coupling strength  $b_3$ , exhibiting a narrow range intermittency transition to GS for linear direct feedback coupling.

the case of the direct feedback coupling configuration can be attributed to the fact that the strength of the coupling  $b_3$  contributes significantly proportional to the strength of the signal  $x_1(t)$  to stabilize all the UPOs immediately at the transition regime after a certain threshold value of the coupling strength as in the case of low-dimensional systems [23].

The robustness of the intermittency transition in a narrow range of the coupling strength  $b_3$  for a range of values of the parameter  $b_2 \in (1.1, 1.6)$  and the delay  $\tau_1 = \tau_2 \in (10, 20)$  is shown in Fig. 12. The three-dimensional plot of the probability of synchronization as a function of the system param-

eter  $b_2$  and the coupling strength  $b_3$  is shown in Fig. 12(a), while Fig. 12(b) shows the three-dimensional plot of  $\Phi(b_3)$  as a function of the coupling delay  $\tau_2$  and the coupling strength  $b_3$ .

**IV. BROAD RANGE INTERMITTENCY TRANSITION TO GS FOR NONLINEAR ERROR FEEDBACK COUPLING OF THE FORM  $f(x_1(t - \tau_2)) - f(x_2(t - \tau_2))$**

Next we demonstrate the existence of the above types of distinct characteristic transitions for nonlinear coupling configurations as well. For this purpose, we consider the unidirectional nonlinear error feedback coupling of the form

$$\dot{x}_1(t) = -ax_1(t) + b_1 f(x_1(t - \tau_1)), \tag{13a}$$

$$\begin{aligned} \dot{x}_2(t) = & -ax_2(t) + b_2 f(x_2(t - \tau_2)) \\ & + b_3 [f(x_1(t - \tau_2)) - f(x_2(t - \tau_2))], \end{aligned} \tag{13b}$$

$$\begin{aligned} \dot{x}_3(t) = & -ax_3(t) + b_2 f(x_3(t - \tau_2)) \\ & + b_3 [f(x_1(t - \tau_2)) - f(x_3(t - \tau_2))], \end{aligned} \tag{13c}$$

where  $f(x)$  is again of the same piecewise linear form as in Eq. (2). The parameters are now fixed as  $a=1.0$ ,  $b_1=b_2=1.2$ , and  $\tau_1=20$  and the coupling delay  $\tau_2=25$ , where the delays alone form the parameter mismatch between the drive and response systems in Eqs. (13a)–(13c). Following Krasovskii-Lyapunov theory, for complete synchronization so that the manifold  $\Delta=x_3(t)-x_2(t)$  between the response  $x_2(t)$  and the auxiliary  $x_3(t)$  approaches zero, one can obtain the stability condition as

$$a > |(b_2 - b_3)f'(x_2(t - \tau_2))|. \tag{14}$$

Consequently from the form of the piecewise linear function (2), the stability condition (14) becomes  $a > |1.5(b_2 - b_3)| > |(b_2 - b_3)|$ . Thus one can take

$$a > |b_2 - b_3| \tag{15}$$

as less stringent condition for (7) to be valid, while

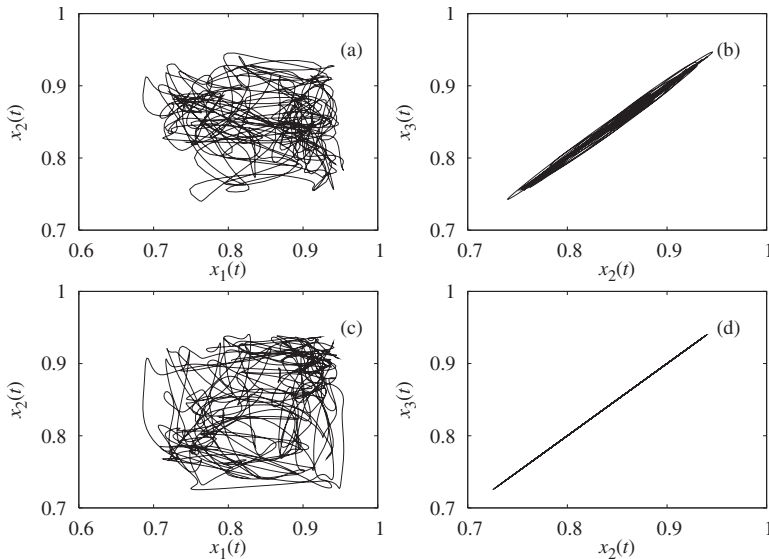


FIG. 13. Dynamics in the phase space of the systems (13a)–(13c). (a) and (b) Approximate GS and CS, respectively, for the value of the coupling strength  $b_3=0.37$ . (c) and (d): Perfect GS and CS, respectively, for the value of the coupling strength  $b_3=0.8$ .



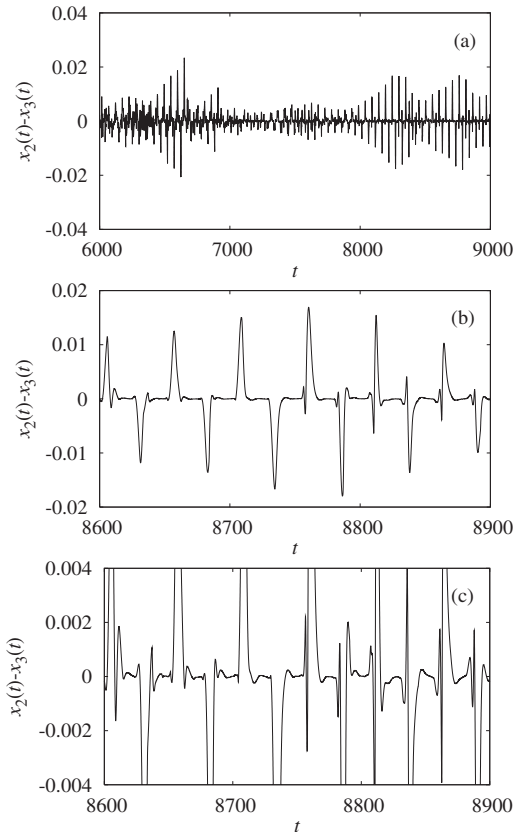


FIG. 14. The intermittent dynamics of the response  $x_2(t)$ , Eq. (13b), and auxiliary  $x_3(t)$ , Eq. (13c), systems for the value of the coupling strength  $b_3=0.37$  for nonlinear error feedback coupling. (a) Time traces of the difference  $x_2(t)-x_3(t)$  corresponding to Fig. 13(b). (b) Enlarged in  $x$  scale to show bursts at periodic intervals when bursts of larger amplitudes  $\Delta > |0.01|$  are considered. (c) Enlarged in  $y$  scale to show random bursts when bursts of smaller amplitudes  $\Delta < |0.01|$  are considered.

$$a > |1.5b_2 - 1.5b_3| \quad (16)$$

can be considered as the most general condition specified by (14) for asymptotic stability of the synchronized state  $\Delta = x_2(t) - x_3(t) = 0$ . For the chosen values of the parameters, the less stringent stability condition (15) is satisfied for the values of the coupling strength  $b_3 \in (0.2, 0.535)$  and the general stability condition (16) is satisfied for  $b_3 > 0.535$ .

As the coupling strength is increased from zero, approximate GS occurs from  $b_3 > 0.2$ . Figure 13(a) shows the approximate GS (IGS) between the drive  $x_1(t)$  [Eq. (13a)] and the response  $x_2(t)$  [Eq. (13b)] systems for the value of the coupling strength  $b_3=0.37$ , while the approximate CS between the response  $x_2(t)$  [Eq. (13b)] and the auxiliary  $x_3(t)$  [Eq. (13c)] systems is shown in Fig. 13(b). Perfect GS and perfect CS are shown in Figs. 13(c) and 13(d), respectively, for  $b_3=0.8$ . The intermittent dynamics exhibited by the coupled systems at the transition regime is shown in Fig. 14, which shows bursts at the period approximately equal to the delay time of the response system  $x_2(t)$  for bursts of amplitude greater than  $|0.01|$  [Fig. 14(b)]. The statistical distribution of the laminar phases away from the intermittent bursts

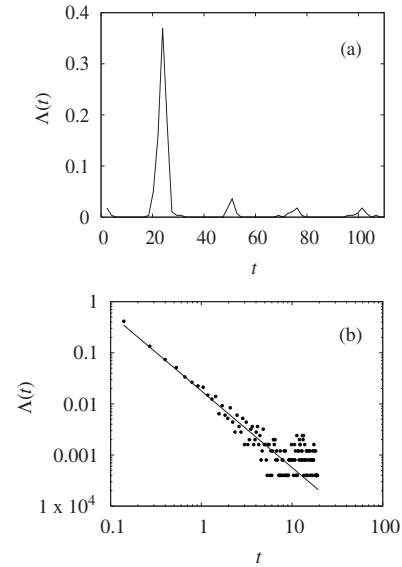


FIG. 15. The statistical distribution of laminar phases corresponding to Fig. 14. (a) For threshold value  $\Delta=|0.1|$  and (b) for threshold value of  $\Delta=|0.0001|$ .

shows an asymptotic  $-1.5$  power law behavior for the threshold value  $\Delta=|0.0001|$  [see Fig. 14(c)], typical for on-off intermittency, which is shown in Fig. 15(b). On the other hand, for the threshold value  $\Delta=|0.1|$  Fig. 15(a) shows periodic structures similar to Fig. 5(a) with peaks occurring approximately at  $t=nT$ ,  $n=1, 2, \dots$ , where  $T \approx \tau_2$  is the period of the lowest periodic orbit of the uncoupled system (13b).

Now, the intermittency transition is again characterized using the probability of synchronization and the sub-Lyapunov exponents as in the previous cases. Figure 16(a) shows the probability of synchronization,  $\Phi(b_3)$ , the value of which remains zero in the range  $b_3 \in (0, 0.2)$  due to the fact that there is lacking any entrainment between the response  $x_2(t)$  and the auxiliary  $x_3(t)$  systems, whereas it fluctuates between the two extreme values in a rather broad range of the coupling strength  $b_3 \in (0.2, 0.42)$ , depicting the existence of an intermittency transition in the corresponding range of  $b_3$ . Perfect CS exists for  $b_3 > 0.42$  as evidenced from the maximum value of  $\Phi(b_3)$ . Correspondingly there exists perfect GS between the drive  $x_1(t)$  and the response  $x_2(t)$  systems. Figure 16(b) shows the transition of sub-Lyapunov exponents of the spectrum of Lyapunov exponents of the coupled drive  $x_1(t)$  [Eq. (13a)] and the response  $x_2(t)$  [Eq. (13b)] systems. The sub-Lyapunov exponents become negative in the range  $b_3 \in (0.2, 0.42)$  confirming the broad range intermittency (IGS) transition in a rather wide range of the coupling strength and this is again due to the gradual stabilization of the unstable periodic orbits of the response systems because of the less significant contribution of the coupling strength  $b_3$  as the error becomes gradually smaller from the transition regime beyond certain threshold value of the coupling strength as discussed in Sec. II. The robustness of the intermittency transition with the system parameter  $b_2$  and the coupling delay  $\tau_2$  as a function of coupling strength  $b_3$  is shown as three-dimensional plots in Fig. 17.

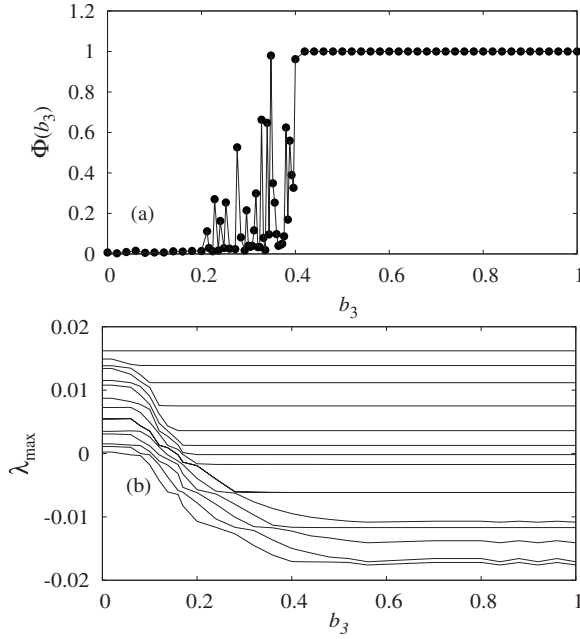


FIG. 16. (a) The probability of synchronization,  $\Phi(b_3)$ , between the response  $x_2(t)$ , Eq. (13b), and the auxiliary  $x_3(t)$ , Eq. (13c), systems. (b) Largest Lyapunov exponents of the coupled drive  $x_1(t)$  and response  $x_2(t)$  systems (13a) and (13b).

**V. NARROW RANGE INTERMITTENCY TRANSITION TO GS FOR NONLINEAR DIRECT FEEDBACK COUPLING OF THE FORM  $f(x_1(t - \tau_2))$**

Now we consider the unidirectional nonlinear coupling of the form

$$\dot{x}_1(t) = -ax_1(t) + b_1 f(x_1(t - \tau_1)), \quad (17a)$$

$$\dot{x}_2(t) = -ax_2(t) + b_2 f(x_2(t - \tau_2)) + b_3 f(x_1(t - \tau_2)), \quad (17b)$$

$$\dot{x}_3(t) = -ax_3(t) + b_2 f(x_3(t - \tau_2)) + b_3 f(x_1(t - \tau_2)). \quad (17c)$$

Choosing the values of the parameters as in the previous case and following a Krasovskii-Lyapunov functional approach for the asymptotically stable synchronized state  $\Delta = x_3(t) - x_2(t) = 0$ , one can obtain the sufficient condition for asymptotic stability for complete synchronization of the response  $x_2(t)$  and the auxiliary  $x_3(t)$  systems as

$$a > |b_2 f'(x_2(t - \tau_2))|. \quad (18)$$

The above stability condition rapidly leads to an immediate transition to the synchronized state even for very feeble values of the coupling strength  $b_3$  for the parameter values satisfying the stability condition (18) as the stability condition is independent of  $b_3$  as in the previous linear coupling case (Sec. III). Hence it is difficult to identify the possible dynamical transitions to synchronized state as a function of the coupling strength  $b_3$ . Hence we study the synchronization

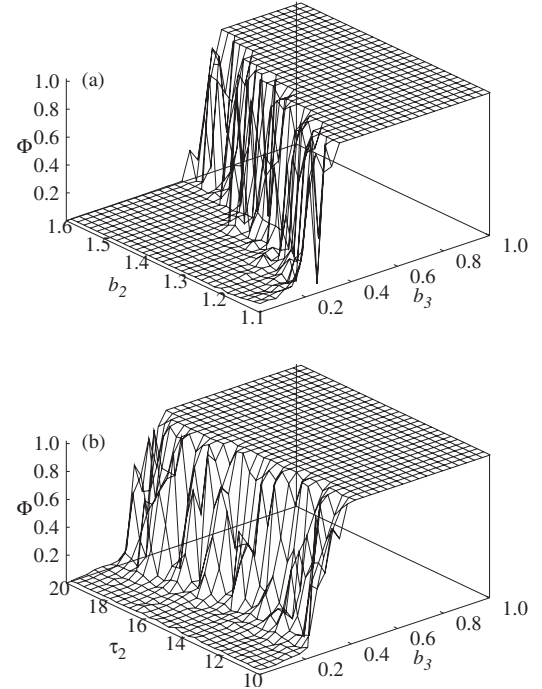


FIG. 17. The probability of synchronization,  $\Phi(b_3)$ , in three-dimensional plots (a) as a function of the system parameter  $b_2$  and the coupling strength  $b_3$  and (b) as a function of the coupling delay  $\tau_2$  and the coupling strength  $b_3$  for the case of the nonlinear error feedback coupling configuration given by Eqs. (13a)–(13c), showing a broad range intermittency transition.

transition as a function of the coupling strength  $b_3$  by choosing the parameters violating the stability condition as  $a = 1.0$ ,  $b_1 = 1.2$ ,  $b_2 = 1.1$ , and  $\tau_1 = \tau_2 = 15$ .

As  $b_3$  is varied from zero for the above values of the parameters, a transition from the desynchronized state to approximate GS occurs for  $b_3 > 0.74$ . The approximate GS (IGS) between the drive  $x_1(t)$  and the response  $x_2(t)$  variables described by Eqs. (17a) and (17b) is shown in Fig. 18(a), whereas Fig. 18(b) shows approximate CS between the response  $x_2(t)$  and auxiliary  $x_3(t)$  variables [Eqs. (17a) and (17c)] for the value of the coupling strength  $b_3 = 0.78$ . Perfect GS and perfect CS are shown in Figs. 18(c) and 18(d), respectively for  $b_3 = 0.9$ . Time traces of the difference  $x_2(t) - x_3(t)$  corresponding to approximate CS [Fig. 18(b)] are shown in Fig. 19, which shows intermittent dynamics with the entrainment of the dynamical variables in random time intervals of finite duration. Figure 19(b) shows an enlarged picture of part of Fig. 19(a). The statistical distribution of the laminar phases again shows a universal asymptotic  $-1.5$  power law behavior for the threshold value  $\Delta = |0.0001|$ , typical for on-off intermittency, as shown in Fig. 20.

As in the previous cases, now we characterize the intermittency transition to GS using the notion of the probability of synchronization,  $\Phi(b_3)$ , and from the changes in the sign of sub-Lyapunov exponents in the spectrum of Lyapunov exponents of the coupled time-delay systems (17a)–(17c). Figure 21(a) shows the probability of synchronization,  $\Phi(b_3)$ , as a function of the coupling strength  $b_3$  calculated from the response  $x_2(t)$  and auxiliary  $x_3(t)$  variables [Eqs.

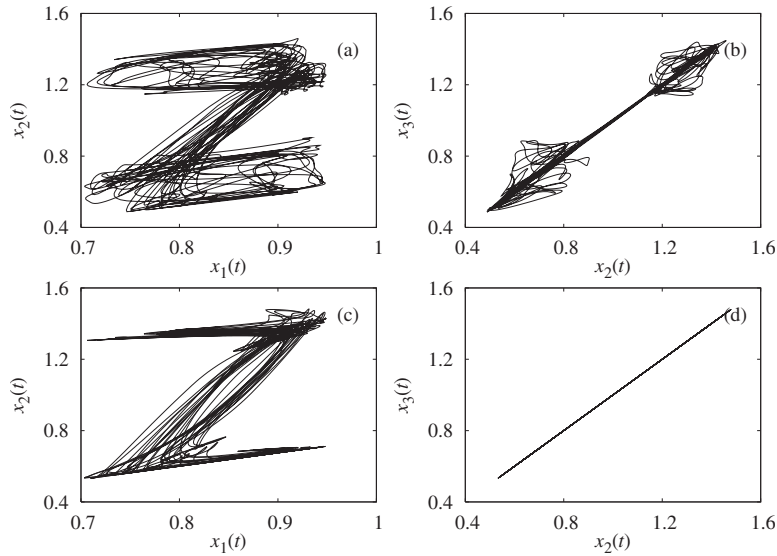


FIG. 18. Dynamics in the phase space of the systems (17a)–(17c). (a) and (b) Approximate GS and CS, respectively, for the value of the coupling strength  $b_3=0.78$ . (c) and (d) Perfect GS and CS, respectively, for the value of the coupling strength  $b_3=0.9$ .

(17b) and (17c)] for CS between them. In the range of  $b_3 \in (0, 0.74)$ , the probability of synchronization remains approximately zero. Upon increasing the value of  $b_3$ ,  $\Phi(b_3)$  oscillates in the narrow range of  $b_3 \in (0.74, 0.78)$  depicting the existence of intermittency transition. This narrow range transition is also confirmed from the transition of successive largest sub-Lyapunov exponents. The spectrum of the first nine largest Lyapunov exponents  $\lambda_{\max}$  of the coupled drive  $x_1(t)$  and response  $x_2(t)$  variables [Eqs. (17a) and (17b)] is shown in Fig. 21(b). It is also evident from the spectrum that the sub-Lyapunov exponents suddenly become negative in the narrow range of  $b_3 \in (0.74, 0.78)$  and then reach saturation values for  $b_3 > 0.78$ . This confirms the narrow range intermittency (IGS) transition to GS. This is also in accordance with the immediate stabilization of all the UPOs of the response system as discussed in Sec. III.

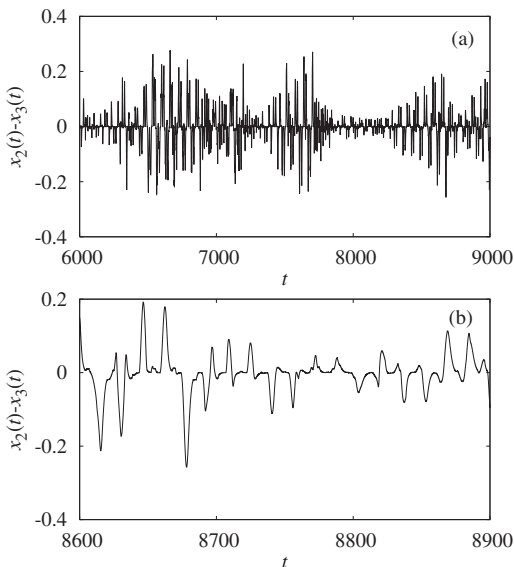


FIG. 19. The intermittent dynamics of the response  $x_2(t)$ , Eq. (17b), and auxiliary  $x_3(t)$ , Eq. (17c), systems for the value of the coupling strength  $b_3=0.78$ . (a) and (b) Time traces of the difference  $x_2(t)-x_3(t)$  corresponding to Fig. 18(b).

The robustness of the transition with the system parameter  $b_2$  and the delay time  $\tau_2$  as a function of coupling strength  $b_3$  is shown as three-dimensional plots in Fig. 22.

## VI. CONCLUSION

In conclusion, we have studied the intermittency transition to generalized synchronization from a desynchronized state in unidirectionally coupled scalar piecewise linear time-delay systems for different coupling configurations using the auxiliary system approach. We have shown that the intermittency transition to GS occurs in a broad range of coupling strength for both linear and nonlinear error feedback coupling configurations whereas it occurs in a narrow range of coupling strength for both the linear and nonlinear direct feedback coupling configurations. It has also been shown that the intermittent dynamics displays periodic intermittent bursts of period equal to the delay time of the response systems in the former cases and it takes place in random time intervals in the latter cases. The robustness of the intermittent dynamics with the system parameters and the delay time is also studied as a function of the coupling strength for both error feedback and direct feedback (linear and nonlinear) coupling configurations. The universality of these intermittent behaviors [36] (periodic and random) and their (broad and narrow range) transitions are also confirmed for different forms of linear and nonlinear coupling configurations in

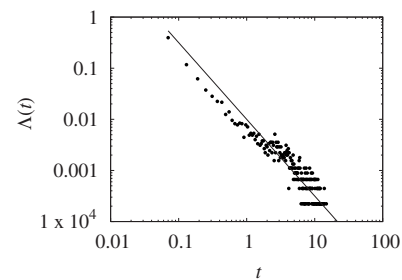


FIG. 20. The statistical distribution of laminar phase for Fig. 19.

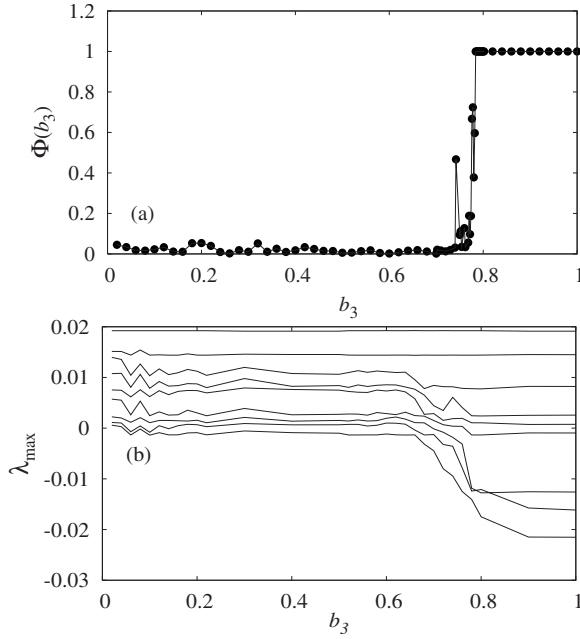


FIG. 21. (a) The probability of synchronization,  $\Phi(b_3)$ , between the response  $x_2(t)$ , Eq. (17b), and the auxiliary  $x_3(t)$ , Eq. (17c), systems. (b) Largest Lyapunov exponents of the coupled drive  $x_1(t)$  and response  $x_2(t)$  systems (17a) and (17b).

other time-delay systems such as Mackey-Glass and Ikeda systems. These distinct characteristic behaviors are analyzed using the analytical stability condition for synchronized state, the probability of synchronization,  $\Phi(b_3)$ , between the response and the auxiliary systems and by the changes in the sign of sub-Lyapunov exponents in the spectrum of Lyapunov exponents of the drive and response systems in both the linear and nonlinearly coupled time-delay systems. In spite of the fact that both the probability of synchronization and the sub-Lyapunov exponents have been calculated from different systems, we have found good agreement between them in showing the intermittency transition in all cases. We hope this study will contribute to the basic understanding of the nature of the transition to GS in coupled time-delay systems and we are now investigating the experimental verification of these findings in nonlinear electronic circuits.

**ACKNOWLEDGMENTS**

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**APPENDIX: STABILITY CONDITION**

To estimate a sufficient condition for the stability of the solution  $\Delta=0$ , we require the derivative of the functional  $V(t)$  along the trajectory of Eq. (4),

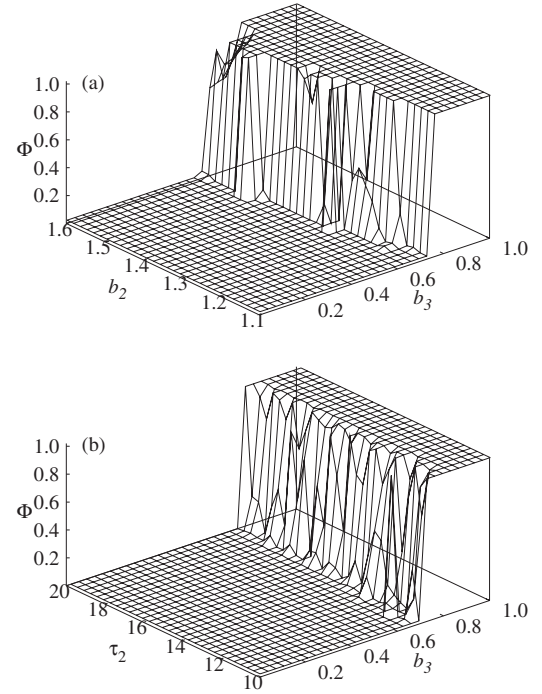


FIG. 22. The probability of synchronization  $\Phi(b_3)$  in three-dimensional plots (a) as a function of the system parameter  $b_2$  and the coupling strength  $b_3$  and (b) as a function of the coupling delay  $\tau_2$  and the coupling strength  $b_3$  for the case of the nonlinear direct feedback coupling configuration given by Eqs. (17a)–(17c) showing a narrow range intermittency transition to GS.

$$\frac{dV}{dt} = -(a + b_3)\Delta^2 + b_2 f'(x_2(t - \tau_2))\Delta\Delta_{\tau_2} + \mu\Delta^2 - \mu\Delta_{\tau_2}^2, \tag{A1}$$

to be negative. The above equation can be rewritten as

$$\frac{dV}{dt} = -\mu\Delta^2\Gamma(X, \mu), \tag{A2}$$

where  $X = \Delta_{\tau_2}/\Delta$ ,  $\Gamma = [(a + b_3 - \mu)/\mu] - [b_2 f'(x_2(t - \tau_2))/\mu]X + X^2$ . In order to show that  $\frac{dV}{dt} < 0$  for all  $\Delta$  and  $\Delta_{\tau_2}$  and so for all  $X$ , it is sufficient to show that  $\Gamma_{\min} > 0$ . One can easily check that the absolute minimum of  $\Gamma$  occurs at  $X = \frac{1}{2\mu} b_2 f'(x_2(t - \tau_2))$  with  $\Gamma_{\min} = [4\mu(a + b_3 - \mu) - b_2^2 f'(x_2(t - \tau_2))^2]/4\mu^2$ . Consequently, we have the condition for stability as

$$a + b_3 > \frac{b_2^2 f'(x_2(t - \tau_2))^2}{4\mu} + \mu = \Phi(\mu). \tag{A3}$$

Again  $\Phi(\mu)$  as a function of  $\mu$  for a given  $f'(x)$  has an absolute minimum at  $\mu = [b_2 f'(x_2(t - \tau_2))]/2$  with  $\Phi_{\min} = |b_2 f'(x_2(t - \tau_2))|$ . Since  $\Phi \geq \Phi_{\min} = |b_2 f'(x_2(t - \tau_2))|$ , from the inequality (A3), it turns out that the sufficient condition for asymptotic stability is

$$a + b_3 > |b_2 f'(x_2(t - \tau_2))|. \tag{A4}$$

Now from the form of the piecewise linear function  $f(x)$  given by Eq. (2), we have

$$|f'(x_2(t - \tau_2))| = \begin{cases} 1.5, & 0.8 \leq |x_2| \leq 4/3, \\ 1.0, & |x_2| < 0.8. \end{cases} \quad (\text{A5})$$

Note that the region  $|x_2| > 4/3$  is outside the dynamics of the present system [see Eq. (2)]. Consequently the stability condition (A4) becomes  $a + b_3 > 1.5|b_2| > |b_2|$ .

Thus one can take  $a + b_3 > |b_2|$  as a less stringent condition for (A4) to be valid, while

$$a > 1.5|b_2|, \quad (\text{A6})$$

as the most general condition specified by (A4) for asymptotic stability of the synchronized state  $\Delta = 0$ .

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